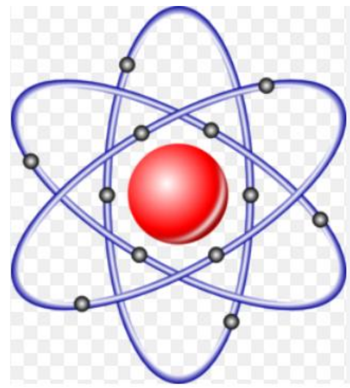
CH X – THE FIVE LAWS OF EXPONENTS



Your chemistry teacher asks you to find the total mass of 3.01×10^{23} electrons assuming that the mass of one election is 9.1×10^{-31} kg. To solve this problem, you need to multiply the two numbers (which is explained in the chapter, *Scientific Notation*). Multiplying 3.01×9.1 is easy; just multiply the decimals. But how do we multiply $10^{23} \times 10^{-31}$? *What do we do with those exponents?* This chapter will answer that question.

❑ *EXPONENTS WITH NUMBERS*

Let's begin by avoiding variables for the moment and just stick to numbers so we can experiment with combining expressions containing exponents. We do this so that we understand, not merely memorize.

I. We want a way to simplify an expression like

$$2^3 \times 2^4$$

Maybe we multiply the 2's and multiply the 3 and the 4. Perhaps we don't multiply the 2's but still multiply the 3 and the 4. Or maybe we need to *add* the 3 and the 4. It's hard to know what to do unless you already know what to do! But we can figure out what to do if we just do the arithmetic:

$$2^3 \times 2^4 = 8 \times 16 = 128 = \mathbf{2^7}$$

From this example, it appears we simply keep the base (the 2) and *add* the exponents. Here's another example where this works:

$$3^2 \cdot 3^3 = 9 \cdot 27 = 243 = \mathbf{3^5}$$

I think we have a neat rule here: *To multiply powers of the same base, just keep the base and add the exponents.* This powerful rule means we can multiply powers without doing any heavy arithmetic; for example:

$$12^9 \times 12^{41} = \mathbf{12^{50}}$$ [Try that the long way!]

Important Note: This shortcut is for multiplying powers of a base; it does not work if the operation is addition. Check this:

$$2^2 + 2^3 = 4 + 8 = 12$$

But 12 is not a nice power of 2 (since 2^3 is 8 and 2^4 is 16).
So if you want to add exponents, be sure that the bases are the same and that the operation is multiplication.

II. Now we study what to do with a "power of a power." For example, we might come across $\left(2^2\right)^3$. Do we add the exponents, or maybe multiply them? Let's just work it out and see:

$$\left(2^2\right)^3 = 4^3 = 64 = 2^6$$

It appears we just multiply the exponents. This means we can simplify something like $\left(5^7\right)^{10}$ by just raising 5 to the product of 7 and 10:

$$\left(5^7\right)^{10} = 5^{70}\text{, and we're done!}$$

III. It's time for a **power of a product**. We can try this:

$$(2 \times 3)^4 = 6^4 = 1{,}296 \qquad \text{(Order of Operations)}$$

But here's another way to get the same result:

$$2^4 \times 3^4 = 16 \times 81 = 1{,}296$$

Do we raise *a product to a power* merely by raising each factor to the power?

IV. Let's try a **quotient of powers**. Here's an example:

$$\frac{10^6}{10^2} = \frac{1{,}000{,}000}{100} = 10{,}000 \qquad \text{(Order of Operations)}$$

Now watch this: If we subtract the exponents and keep the base of 10, we get 10^4, which is also equal to 10,000.

Let's do a second example where we put the bigger exponent on the bottom:

$$\frac{2^3}{2^7} = \frac{8}{128} = \frac{1}{16}$$

But $\frac{1}{16}$ can also be written as $\frac{1}{2^4}$, where the 4 results from subtracting the exponents. There's something going on here.

V. Our fifth example in this section will look at a **power of a quotient**.

$$\left(\frac{2}{3}\right)^4 = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{16}{81} \qquad \text{(the meaning of exponent)}$$

But if we just raise 2 to the 4th power, and then raise 3 to the 4th power, and write one over the other, we get the same result:

$$\frac{2^4}{3^4} = \frac{16}{81}$$

❑ EXPONENTS WITH VARIABLES

For each of the following five examples, we will "stretch-and-squish," and then we'll generalize what we observe to the official ***Five Laws of Exponents***.

I. We start by finding the product of x^3 and x^4:

$$x^3x^4 = (x\,x\,x)(x\,x\,x\,x) = x\,x\,x\,x\,x\,x\,x = x^7$$

Notice that the bases (the *x*'s) are the same, and it's a multiplication problem. As long as the bases are the same, and it's a multiplication problem, it appears that we merely need to write down the base, and then add the exponents together to get the exponent of the answer. That is, $x^ax^b = x^{a+b}$.

$$x^3x^4 = x^7$$

[This law of exponents, where we multiply powers of the same base by adding the exponents, is almost always called the First Law of Exponents. The four laws that follow are not in any particular order.]

II. For our second example, let's raise a power to a power:

$$(x^4)^2 = (x\,x\,x\,x)^2 = (x\,x\,x\,x)(x\,x\,x\,x) = x\,x\,x\,x\,x\,x\,x\,x = x^8$$

We appear to have a shortcut at hand. Merely multiply the two exponents together and we're done. So, to raise a power to a power, we can write a general rule: $(x^a)^b = x^{ab}$.

$$(x^4)^2 = x^8$$

III. Now we're to try raising a product to a power; for instance,

$$(ab)^5 = (ab)(ab)(ab)(ab)(ab) = (aaaaa)(bbbbb) = a^5b^5$$

In general, when raising a product to a power, raise each factor to the power: $(xy)^n = x^n y^n$.

$$(ab)^5 = a^5b^5$$

Note that the quantity in the parentheses is a single term — there's no adding or subtracting in the parentheses. In fact, if there are two or more terms in the parentheses, this law of exponents does not apply.

IV. Next we divide powers of the same base. We'll need two examples for this law of exponents.

A. $$\frac{x^6}{x^2} = \frac{xxxxxx}{xx} = \frac{\cancel{x}\cancel{x}xxxx}{\cancel{x}\cancel{x}} = x^4$$

B. $$\frac{y^4}{y^6} = \frac{yyyy}{yyyyyy} = \frac{\cancel{y}\cancel{y}\cancel{y}\cancel{y}}{\cancel{y}\cancel{y}\cancel{y}\cancel{y}yy} = \frac{1}{y^2}$$

$$\frac{x^6}{x^2} = x^4 \qquad \frac{y^4}{y^6} = \frac{1}{y^2}$$

In general, when dividing powers of the same base, subtract the exponents, leaving the remaining factors on the top if the top exponent is bigger, and on the bottom if the bottom exponent is bigger.

V. Our last example in this section is the process of raising a quotient to a power. As usual, we stretch and squish; then we generalize to a law of exponents.

$$\left(\frac{a}{b}\right)^4 = \left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right) = \frac{aaaa}{bbbb} = \frac{a^4}{b^4}$$

$$\left(\frac{a}{b}\right)^4 = \frac{a^4}{b^4}$$

In general, we can raise a quotient to a power by raising both the top and bottom to the power: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

❑ SUMMARY OF THE FIVE LAWS OF EXPONENTS

Exponent Law	Example
$x^a x^b = x^{a+b}$	$x^2 x^6 = x^8$
$(x^a)^b = x^{ab}$	$(a^4)^3 = a^{12}$
$(xy)^n = x^n y^n$	$(wz)^7 = w^7 z^7$
For $a > b$, $\dfrac{x^a}{x^b} = x^{a-b}$ For $b > a$, $\dfrac{x^a}{x^b} = \dfrac{1}{x^{b-a}}$	$\dfrac{x^{10}}{x^2} = x^8$ $\dfrac{a^3}{a^7} = \dfrac{1}{a^4}$
$\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$	$\left(\dfrac{a}{b}\right)^5 = \dfrac{a^5}{b^5}$

❑ SINGLE-STEP EXAMPLES

EXAMPLE 1 [Let's call this law the *First Law of Exponents*.]

A. $A^7 A^5 = A^{7+5} = A^{12}$

The bases are the same, and it's a multiplication problem. So we repeat the base and add the exponents.

B. $x^2 x^3 x^4 = x^{2+3+4} = x^9$

All the bases are the same, and it's a multiplication problem, and so we add the exponents.

C. $(x+y)^4(x+y)^9 = (x+y)^{13}$

It doesn't matter what the base is, as long as we're multiplying powers of the same base.

EXAMPLE 2:

A. $(c^{10})^2 = c^{20}$

Raising a base to a power, and then raising that result to a further power requires simply that we multiply the exponents.

B. $\left((x^2)^3\right)^4 = x^{24}$

$(x^2)^3 = x^6$, and then $(x^6)^4 = x^{24}$
Shortcut: Just multiply all three exponents.

EXAMPLE 3:

A. $(ax)^5 = a^5x^5$

It's a power of a product (a single term). So just raise each factor to the 5th power.

B. $(abc)^7 = a^7b^7c^7$

Even a term with three factors can be raised to the 7th power by raising each factor to the 7th power.

EXAMPLE 4:

A. $\dfrac{x^7}{x^5} = x^2$

Since 7 > 5, we divide powers of the same base by subtracting the exponents.

B. $\dfrac{w^{15}}{w^{25}} = \dfrac{1}{w^{10}}$

Since the bigger exponent is on the bottom, we subtract 15 from 25 and leave that power of w on the bottom.

EXAMPLE 5:

A. $\left[\dfrac{x}{z}\right]^7 = \dfrac{x^7}{z^7}$

To raise a quotient to a power, just raise both the top and bottom to the 7th power.

B. $\left(\dfrac{a+b}{u-w}\right)^{23} = \dfrac{(a+b)^{23}}{(u-w)^{23}}$

Just raise top and bottom to the 23rd power.

Homework

1. Use the Five Laws of Exponents to simplify each expression:

a. a^3a^4 b. $x^5x^6x^2$ c. y^3y^3 d. $z^{12}z$

e. $(x^3)^4$ f. $(z^8)^2$ g. $(n^{10})^{10}$ h. $(a^1)^7$

i. $(ab)^3$ j. $(xyz)^5$ k. $(RT)^1$ l. $(math)^5$

m. $\dfrac{a^8}{a^2}$ n. $\dfrac{b^3}{b^9}$ o. $\dfrac{w^5}{w^5}$ p. $\dfrac{Q^{100}}{Q^{50}}$

q. $\left(\dfrac{k}{w}\right)^4$ r. $\left(\dfrac{a}{b}\right)^{99}$ s. $\left(\dfrac{1}{m}\right)^{20}$ t. $a(bc)^2$

2. Use the Five Laws of Exponents to simplify each expression:

a. a^3a^5 b. $u^5u^7u^2$ c. $y^{30}y^{30}$ d. $z^{14}z$

e. $(x^4)^5$ f. $(z^9)^2$ g. $(n^{100})^{10}$ h. $(a^1)^9$

i. $(xy)^4$ j. $(abc)^{17}$ k. $(pn)^1$ l. $(love)^4$

m. $\dfrac{a^{10}}{a^2}$ n. $\dfrac{b^3}{b^{12}}$ o. $\dfrac{w^9}{w^9}$ p. $\dfrac{Q^{100}}{Q^{20}}$

q. $\left(\dfrac{x}{w}\right)^3$ r. $\left(\dfrac{a}{b}\right)^{999}$ s. $\left(\dfrac{1}{z}\right)^{22}$ t. $w(xy)^3$

❑ *When NOT to Use the Five Laws of Exponents*

$\boldsymbol{a^5b^6}$ cannot be simplified. Although the First Law of Exponents demands that the expressions be multiplied — and they are — it also requires that the bases be the same — and they aren't.

$\boldsymbol{x^3 + x^4}$ cannot be simplified. Even though the bases are the same, the First Law of Exponents requires that the two powers of x be multiplied.

$\boldsymbol{w^3 + w^3}$ can be simplified, but not by the First Law of Exponents, since the powers of w are not being multiplied. But the two terms are *like terms*, which means we add them together to get $\boldsymbol{2w^3}$.

$\boldsymbol{(a + b)^{23}}$ does not equal $a^{23} + b^{23}$. You may think that the third law of exponents, $(xy)^n = x^n y^n$, might apply, but it does not, and that's because xy is a single term, whereas $a + b$ consists of two terms. You'll have to wait until the chapter entitled *The Binomial Theorem* to learn a clever way to calculate the 23rd power of $a + b$. Also, you may have already learned in this class that $(a + b)^2$ is actually equal to $a^2 + 2ab + b^2$, and so again, $(a + b)^n \neq a^n + b^n$ (for $n \geq 2$).

Homework

3. Simplify each expression:

a. y^4y^4	b. a^3b^4	c. $x^4x^3x^2$	d. $p^3t^2p^2$
e. $a^3 + a^5$	f. a^3a^5	g. $n^4 + n^4$	h. $x^3 - x^3$
i. $(x + y)^{55}$	j. $Q^2 + Q^2$	k. u^5w^6	l. $h^6 - h^2$
m. $(a - b)^2$	n. $(ab)^2$	o. $(x^3)^3$	p. $x^4 + x^5$
q. $x^{14} + x^{14}$	r. $y^{12} - y^{12}$	s. $a^8 + a^9$	t. $a^{10} + a^{10}$
u. $(xy)^2$	v. $(x + y)^2$	w. a^3b^4	x. $a^3 + b^4$
y. $a(a^2)(b^2)b$	z. $n^6 + n^6$		

❑ *MULTI-STEP EXAMPLES*

EXAMPLE 6:

A. $(-3x^2y^3)(-5xy^7) = (-3)(-5)\,(x^2x)(y^3y^7) = 15x^3y^{10}$

B. $-2x^2y(xy - 4x^3y^4) = -2x^3y^2 + 8x^5y^5$ (distribute)

C. $\left(2a^2b^3\right)^4 = 2^4(a^2)^4(b^3)^4 = 16a^8b^{12}$ (the 2 is in the parentheses)

D. $7\left(xy^{10}\right)^5 = 7x^5(y^{10})^5 = 7x^5y^{50}$ (the 7 is not in the parentheses)

E. $\left(\frac{a^2}{b^3}\right)^7 = \frac{(a^2)^7}{(b^3)^7} = \frac{a^{14}}{b^{21}}$

F. $\left(\frac{x^3y^9}{xy^{12}}\right)^5 = \left(\frac{x^2}{y^3}\right)^5 = \frac{(x^2)^5}{(y^3)^5} = \frac{x^{10}}{y^{15}}$ (simplify the inside first)

Homework

4. Simplify each expression:

a. $(-5a^3b^4)(-2a^2b)$

b. $(7xy)(-7x^2y^5)$

c. $(-2uw)(2uw)$

d. $x^3(2x^2 - x - 1)$

e. $3y^2(3y^2 - y + 3)$

f. $\left(a^2b^3\right)^4$

g. $\left(-5m^3n^{10}\right)^3$

h. $\left[-3p^3q^3\right]^4$

i. $4\left(xy^7\right)^{10}$

j. $-10(-2c^3y^4)^3$

k. $\left(\frac{a^3}{c^2}\right)^{10}$

l. $\left[\frac{2x^3}{3xy^4}\right]^3$

m. $\left(\dfrac{a^2b^3}{a^4b}\right)^5$ n. $2(3x^2y^3)^4$

❑ *ZERO AS AN EXPONENT*

You may have already learned that anything to the zero power is **1** (as long as it's not zero to the zero). Now that we have some laws of exponents at our disposal, we can actually prove this fact; that is, we can prove that $\boldsymbol{x^0 = 1}$.

Consider the fraction $\dfrac{x^6}{x^6}$, where $x \neq 0$. On the one hand, anything (except zero) divided by itself is 1:

$$\frac{x^6}{x^6} = \underline{1}$$

On the other hand, by one of the exponent laws, since it's division of powers of the same base, we can **subtract** the exponents:

$$\frac{x^6}{x^6} = \underline{x^0}$$

See the logic? Since $\dfrac{x^6}{x^6}$ is equal to 1 and it's also equal to x^0, it must follow that

$$\boldsymbol{x^0 = 1}$$

> Any number (except 0) raised to the 0 power is 1.

<u>EXAMPLE 7:</u>

A. $(x - 3y + z)^0 = \mathbf{1}$ (any quantity ($\neq 0$) to the zero power is 1)

B. $(abc)^0 = \mathbf{1}$ (any quantity ($\neq 0$) to the zero power is 1)

C. $a + b^0 = \boldsymbol{a + 1}$ (the exponent is on the b only)

D. $uw^0 = u(1) = \boldsymbol{u}$ (the exponent is on the w only)

E. $(-187)^0 = \mathbf{1}$ (the exponent is on the –187)

F. $-14^0 = \mathbf{-1}$ (the exponent in on the 14, not on the minus sign)

Homework

5. Evaluate each expression:

a. $2^0 + 3^0$ b. $4^0 \cdot 5^0$ c. $6^0 + 2^1 + 2^2 + 2^3 + 2^4$

d. $(1 + 12)^0$ e. $2^5 - 2^3 + 2^1 - 2^0$ f. $(8 - 5)^0 + (10 - 8)^1$

g. $2^0 \times 2^1 \times 2^2 \times 2^3 \times 2^4$

h. $\left(\frac{12}{29}\right)^0 + \left(\frac{100}{77}\right)^0 - (20 - \pi - 3)^0 + \left(3^2 - 7\right)^1$

6. Simplify each expression:

a. x^0 b. xy^0 c. $x + y^0$ d. $(x + y)^0$

e. $(ab)^0$ f. $\left(\frac{a^2}{b^3}\right)^0$ g. $\frac{(x^2)^0}{y^3}$ h. m^0m

i. $x^0 + x^0$ j. Q^0Q^0 k. $a^0 - a^0$ l. $\left(\frac{2x^2y^0}{-3ab^{10}}\right)^0$

❑ *FINAL NOTE*

The Introduction to this chapter talked about working the following problem:

$$10^{23} \times 10^{-31}$$

The bases are the same (10) and the operation is multiplication. Thus, the First Law of Expoents says we can bring the 10 along and add the exponents:

10^{-8} (which equals 0.00000001)

Review Problems

7. Simplify: $\left(-3x^3y^4x^7\right)^3$

8. Simplify: $-3\left(x^5x^4x^8\right)^3$

9. Simplify: $\dfrac{a^2b^3c^9}{ab^4c^3}$

10. Simplify: $-2y^3(3y^4 - 2y^3 + 1)$

11. Simplify: $x^{12} + x^{14}$

12. Simplify: $u^{22} + u^{22}$

13. Simplify: $abcd^0e^0$

14. Simplify: $\left[\dfrac{10^0a^0b^{14}}{a^3b^7}\right]^5$

Solutions

1.

a. a^7	b. x^{13}	c. y^6	d. z^{13}
e. x^{12}	f. z^{16}	g. n^{100}	h. a^7
i. a^3b^3	j. $x^5y^5z^5$	k. RT	l. $m^5a^5t^5h^5$
m. a^6	n. $\frac{1}{b^6}$	o. 1	p. Q^{50}
q. $\frac{k^4}{w^4}$	r. $\frac{a^{99}}{b^{99}}$	s. $\frac{1}{m^{20}}$	t. ab^2c^2

2.

a. a^8	b. u^{14}	c. y^{60}	d. z^{15}
e. x^{20}	f. z^{18}	g. n^{1000}	h. a^9
i. x^4y^4	j. $a^{17}b^{17}c^{17}$	k. pn	l. $l^4o^4v^4e^4$
m. a^8	n. $\frac{1}{b^9}$	o. 1	p. Q^{80}
q. $\frac{x^3}{w^3}$	r. $\frac{a^{999}}{b^{999}}$	s. $\frac{1}{z^{22}}$	t. wx^3y^3

3.

a. y^8	b. As is	c. x^9	d. p^5t^2
e. As is	f. a^8	g. $2n^4$	h. 0
i. As is (for now)	j. $2Q^2$	k. As is	l. As is
m. $a^2 - 2ab + b^2$	n. a^2b^2	o. x^9	p. As is
q. $2x^{14}$	r. 0	s. As is	t. $2a^{10}$
u. x^2y^2	v. $x^2 + 2xy + y^2$		w. As is
x. As is	y. a^3b^3	z. $2n^6$	

4.

a. $10a^5b^5$	b. $-49x^3y^6$	c. $-4u^2w^2$	d. $2x^5 - x^4 - x^3$
e. $9y^4 - 3y^3 + 9y^2$		f. a^8b^{12}	g. $-125m^9n^{30}$
h. $81p^{12}q^{12}$	i. $4x^{10}y^{70}$	j. $80c^9y^{12}$	k. $\frac{a^{30}}{c^{20}}$

l. $\dfrac{8x^6}{27y^{12}}$ m. $\dfrac{b^{10}}{a^{10}}$ n. $162x^8y^{12}$

5. a. 2 b. 1 c. 31 d. 1
e. 25 f. 3 g. 1024 h. 3

6. a. 1 b. x c. $x+1$ d. 1 e. 1 f. 1
g. $\dfrac{1}{y^3}$ h. m i. 2 j. 1 k. 0 l. 1

7. $-27x^{30}y^{12}$

8. $-3x^{51}$

9. $\dfrac{ac^6}{b}$

10. $-6y^7 + 4y^6 - 2y^3$

11. As is

12. $2u^{22}$

13. abc

14. $\dfrac{b^{35}}{a^{15}}$

I have no particular talent.
I am merely inquisitive.

Albert Einstein